GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES

ENRESDOWEDNESS OF TYPE - III UNICYCLIC GRAPHS

Dr. P. Sumathi*[1] & R. Esther Felicia[2]

[*1]Associate Professor, Department of Mathematics, C. Kandaswami Naidu College for Men, Chennai
[2]Ph.d Research Scholar, Assistant Professor, Department of Mathematics, ShriKrishnaswamy College for Women, Chennai

ABSTRACT

Let G = (V, E) be a non empty, finite, simple graph. A dominating set of a graph G containing a minimum dominating set of G is called a γ - endowed dominating set of G. If that set is of cardinality k then it is called a $k\gamma$ – endowed dominating set. k - $\gamma_r$enresdowed graph is one in which every restrained dominating set of cardinality k contains a minimum restrained dominating set. Consider a cycle graph G, in which a set of different paths is attached to every vertex of the cycle. In this paper, the enresdowedness property for the unicyclic graphs with a set of different paths attached to every vertex of the cycle is obtained.

Keywords: *Enresdowed graphs, Unicyclic graphs.*

I. INTRODUCTION

Let G = (V, E) be a non empty, finite, simple graph. A subset D of V(G) is called a dominating set of G if for every $v \in V - D$, there exists $u \in D$ such that u and v are adjacent. The minimum cardinality of the dominating set is called the domination number and it is denoted by $\gamma(G)$. The restrained dominating set of a graph is a dominating set in which every vertex in V – D is adjacent to some other vertex in V – D[6]. The minimum cardinality of the restrained dominating set is called the restrained domination number and it is denoted by $\gamma_r(G)$. A graph is said to be k - γ_r enresdowed graph if every restrained dominating set of cardinality k contains a minimum restrained dominating set. Consider a unicyclic graph G which contains a cycle C_n , $n\geq 3$, and a set of paths P_t, $t\geq 2$, where these set s, $s\geq 1$of different paths are attached to each vertex of C_n.Anders Sune Pedersen,PrebenDahlVestergaardobtained the upper and lower bounds for the number of independent sets in a unicyclic graph in terms of its order[1].A unicyclic graph is a connected graph with a unique cycle. A unicyclic graph is called fully loaded if every vertex on its unique cycle has degree at least three[7]. Joanna Raczek characterize all connected unicyclic graphs with the domination multisubdivision number equal to three[2].

II. RESULTS ON TYPE – III UNICYCLIC ENRESDOWED GRAPHS

Definition 2.1

Let k be a positive integer. A simple, finite, non trivial graph G = (V, E) is called a k - γ_r enresdowed graph if every restrained dominating set of G of cardinality k contains a minimum restrained dominating set γ_r of G.[5]

Definition 2.2

Let G be a unicyclicgraph C_nP_t , for $n\geq 3$, $t\geq 2$. Let $\{v_i\}, 1 \leq i \leq n$ be the set of vertices of C_n. The graph G contains a set of n copies of distinct paths $\{P_{it_j}\}, 1 \leq i \leq n$ and $2 \leq j \leq s_i$, which are attached to each vertex $\{v_i\}, 1 \leq i \leq n$ of the cycle C_n , for $n\geq 3$. The set of vertices $\{v_i\}, 1 \leq i \leq n$ is considered as the initial vertex for the set of all paths $\{P_{it_j}\} 1 \leq i \leq n, 2 \leq j \leq s_i$ attached to each $\{v_i\}, 1 \leq i \leq n$.

Theorem 2.3

Let G be a unicyclicgraph C_nP_t , for $n\geq 3$ and $t\geq 2$. Let $\{v_i\}, 1 \leq i \leq n$ be the set of vertices of C_n. The graph G contains a set of n copies of distinct paths $\{P_{it_j}\}$, $1 \leq i \leq n$ and $2 \leq j \leq S_i$, which are attached to each vertex $\{v_i\}$,

$1 \leq i \leq n$ of the cycle C_n , for $n \geq 3$, such that the cardinality of any path P_{it_j}, $1 \leq i \leq n$, $2 \leq j \leq S_i$ is not same as any other path $P_{it_{j+1}}$, for $1 \leq i \leq n$, $2 \leq j \leq S_i$ which are attached to same v_i, $1 \leq i \leq n$ in G. The set of vertices $\{v_i\}$,$1 \leq i \leq n$ of the cycle C_n , for $n \geq 3$ is considered as the initial vertices for the set of all paths $\{P_{it_j}\}$,$1 \leq i \leq n$, $2 \leq j \leq S_i$, attached to it. Let D be the minimum restrained dominating set of G, then G is k - γ_r enresdowed for any k , where $\gamma_r \leq k \leq n + \left| \bigcup^{n}_{\substack{i=1 \\ 2 \leq j \leq S_i}} P_{it_j} - S_i \right|$

Proof

Given G is a unicyclicgraph C_nP_t , for $n \geq 3$ and $t \geq 2$. Let $\{v_1,v_2, v_3,\ldots,v_i,\ldots,v_n\}$, $1 \leq i \leq n$, be the set of all vertices of the cycle C_n, $n \geq 3$. Let P_{1t_2} be a path P_2, which consist of the vertex set $\{u_{1,21} ,u_{1,22}\}$, such that the vertex $u_{1,21} = v_1$, P_{1t_3} be a path P_3, with the vertex set $\{u_{1,31} ,u_{1,32} , u_{1,33}\}$, where the vertex $u_{1,31} = v_1$, P_{1t_4} be a path P_4, with the vertex set $\{u_{1,41} ,u_{1,42} , u_{1,43} ,u_{1,44}\}$ such that the vertex $u_{1,41} = v_1$. Without loss of generality, consider any path P_{1t_l}, where P_{1t_l} be a path P_l, $2 \leq l \leq S_1$, which contains the vertex set $\{u_{1,l1} ,u_{1,l2} , u_{1,l3} ,\ldots,u_{1,ll}\}$ where the vertex $u_{1,l1} = v_1$. Proceeding similarly, consider any path $P_{1t_{S_1}}$, where the path $P_{1t_{S_1}}$ is a path P_{S_1}which consist of the vertex set $\{u_{1,S_11} ,u_{1,S_12} , u_{1,S_13} ,\ldots,u_{1,S_1S_1}\}$, such that the vertex $u_{1,S_11} = v_1$. Thus the set of vertices $\{u_{1,21} ,u_{1,22} , u_{1,31} ,u_{1,32} , u_{1,33}, u_{1,41} ,u_{1,42} , u_{1,43} ,u_{1,44},\ldots, u_{1,l1} ,u_{1,l2} , u_{1,l3} ,\ldots,u_{1,ll},\ldots,u_{1,S_11} ,u_{1,S_12} , u_{1,S_13} ,\ldots,u_{1,S_1S_1}\}$, belongs to the paths P_{1t_2},P_{1t_3},P_{1t_4},……,P_{1t_l},……., $P_{1t_{S_1}}$ of G, where these sets of all paths are attached to the vertex v_1 of C_n.

Without loss of generality, consider another new set of paths P_{2t_2},P_{2t_3},P_{2t_4},…,P_{2t_l},…., $P_{2t_{S_2}}$ which are attached to the vertex v_2 of C_n for $n \geq 3$. Let the path P_{2t_2} be a path P_2, with the vertex set $\{u_{2,21} ,u_{2,22}\}$, where the vertex $u_{2,21} = v_2$ and P_{2t_3} be a path P_3, which consist of the vertex set $\{u_{2,31} ,u_{2,32} , u_{2,33}\}$, such that the vertex $u_{2,31} = v_2$. Let P_{2t_4} be a path P_4 with the vertex set $\{u_{2,41} ,u_{2,42}, u_{2,43} ,u_{2,44}\}$ where the vertex $u_{2,41} = v_2$. Without loss of generality consider any path P_{2t_l} where P_{2t_l} be a path P_l, $2 \leq l \leq S_2$, with the set of vertices $\{u_{2,l1} ,u_{2,l2} , u_{2,l3} ,\ldots,u_{2,ll}\}$, such that the vertex $u_{2,l1} = v_2$. Finally consider any path $P_{2t_{S_2}}$, where the path $P_{2t_{S_2}}$ is a path P_{S_2} which consist of the vertex set $\{u_{2,S_21} ,u_{2,S_22} , u_{2,S_23} ,\ldots,u_{2,S_2S_2}\}$, such that the vertex $u_{2,S_21} = v_2$. Thus the set of vertices $\{u_{2,21} ,u_{2,22},u_{2,31} ,u_{2,32} , u_{2,33},u_{2,41} ,u_{2,42} , u_{2,43} ,u_{2,44},\ldots,u_{2,l1} ,u_{2,l2} , u_{2,l3} ,\ldots,u_{2,ll},\ldots,u_{2,S_21} ,u_{2,S_22} , u_{2,S_23} ,\ldots,u_{2,S_2S_2}\}$ which belongs to the set of paths $\{P_{it_j}\}$, for $i = 2$, and $2 \leq j \leq S_2$.

In general, consider a new set of paths, P_{it_2},P_{it_3},P_{it_4},……,P_{it_l}, …., $P_{it_{S_i}}$, $1 \leq i \leq n$. These paths are attached to the vertex v_i, $1 \leq i \leq n$ of C_n for $n \geq 3$. Let P_{it_2} be a path P_2, with the vertex set $\{u_{i,21} ,u_{i,22}\}$, such that the vertex $u_{i,21} = v_i$, $1 \leq i \leq n$. The path P_{it_3} be a path P_3 which consist of the set of vertices $\{u_{i,31} ,u_{i,32} , u_{i,33}\}$,where the vertex $u_{i,31} = v_i$, $1 \leq i \leq n$. Let P_{it_4} be a path P_4 with the vertex set $\{u_{i,41} ,u_{i,42}, u_{i,43} ,u_{i,44}\}$, such that the vertex $u_{i,41} = v_i$, $1 \leq i \leq n$. Similarly consider any path P_{it_l}, where P_{it_l} is a path P_l, $2 \leq l \leq S_i$, $1 \leq i \leq n$ which consist of the set of vertices $\{u_{i,l1} ,u_{i,l2} , u_{i,l3} , \ldots,u_{i,ll}\}$, such that the vertex $u_{i,l1} = v_i$, $1 \leq i \leq n$. Finally consider any path $P_{it_{S_i}}$, $1 \leq i \leq n$,where the path $P_{it_{S_i}}$ is a path P_{S_i} which consist of the vertex set $\{u_{i,S_i1} ,u_{i,S_i2} , u_{i,S_i3} ,\ldots,u_{i,S_iS_i}\}$, for $1 \leq i \leq n$, such that the vertex $u_{i,S_i1} = v_i$, $1 \leq i \leq n$. Then the set of vertices $\{u_{i,21} ,u_{i,22}, u_{i,31} ,u_{i,32} , u_{i,33},u_{i,41} ,u_{i,42} , u_{i,43} ,u_{i,44},\ldots,u_{i,l1} ,u_{i,l2} , u_{i,l3} ,\ldots,u_{i,ll},\ldots,u_{i,S_i1} ,u_{i,S_i2} , u_{i,S_i3} , \ldots,u_{i,S_iS_i}\}$ belongs to the set of paths P_{it_2},P_{it_3},P_{it_4},……,P_{it_l},…., $P_{it_{S_i}}$, $1 \leq i \leq n$.

Proceeding similarly, consider a new set of paths in G. Let P_{nt_2}, be a path P_2, with the vertex set $\{u_{n,21} ,u_{n,22}\}$, where the vertex $u_{n,21} = v_n$. The path P_{nt_3}, be a path P_3, which consist of the set of vertices $\{u_{n,31} ,u_{n,32} , u_{n,33}\}$such that the vertex $u_{n,31} = v_n$. Let P_{nt_4} be a path P_4, with the vertex set $\{u_{n,41} ,u_{n,42} , u_{n,43} ,u_{n,44}\}$, where the vertex $u_{n,41} = v_n$. Proceeding similarly, consider the path P_{nt_l}, where P_{nt_l}, $2 \leq l \leq S_n$ is a path P_l which contains the set of vertices $\{u_{n,l1} ,u_{n,l2} , u_{n,l3} ,\ldots,u_{n,ll}\}$, such that the vertex $u_{n,l1} = v_n$. Finally, consider the path $P_{nt_{S_n}}$, where the path $P_{nt_{S_n}}$ is a path P_{S_n} with the vertex set $\{u_{n,S_n1}, u_{n,S_n2}, u_{n,S_n3}, \ldots, u_{n,S_nS_n}\}$, where the vertex $u_{n,S_n1} = v_n$. Then the set of vertices $\{u_{n,21}, u_{n,22}, u_{n,31}, u_{n,32}, u_{n,33}, u_{n,41}, u_{n,42}, u_{n,43}, u_{n,44}, \ldots, u_{n,l1}, u_{n,l2}, u_{n,l3}, \ldots, u_{n,ll}, \ldots, u_{n,S_n1}, u_{n,S_n2}, u_{n,S_n3}, \ldots, u_{n,S_nS_n}\}$, belongs to the set of paths $P_{nt_2}, P_{nt_3}, P_{nt_4}, \ldots, P_{nt_l}, \ldots, P_{nt_{S_n}}$, where these set of paths are attached to the vertex v_n of C_n. Thus the graph $G = C_nP_t$, for $n \geq 3$ and $t \geq 2$ is obtained.

Let D be the minimum restrained dominating set of G. The set of all paths $\{P_{it_j}\}$, $1 \leq i \leq n$ and $2 \leq j \leq S_i$ will be of any one of the following types.

Case (i) Suppose if $P_{it_l} = P_{3m-1}$, for $m \geq 1$, $1 \leq i \leq n$, where $l = 2,5,8,\ldots$, then the path P_{it_l} be P_l, where the vertex set of P_l, be $\{u_{i,l1}, u_{i,l2}, u_{i,l3}, \ldots, u_{i,ll}\}$ for $1 \leq i \leq n$, such that the vertex $u_{i,l1} = v_i$. Without loss of generality, choose the set of vertices $u_{i,l2}$, for $1 \leq i \leq n$, $l = 2,5,8,\ldots$ for the γ_r set D from the path of the type P_{3m-1}, for $m \geq 1$, from each u_i then the set of vertices $u_{i,l3}$, for $l \neq 2$ where $l = 5,8,\ldots$ are dominated. Similarly choose the set of vertices $u_{i,l5}$, for $l = 5,8,\ldots$ for the γ_r set D, then the set of vertices $u_{i,l4}$, $l = 5,8,\ldots$ and $u_{i,l6}$, $1 \leq i \leq n$, $l = 8,\ldots$ are dominated, such that the set of vertices $u_{i,l3}$ and $u_{i,l4}$ $l = 5,8,\ldots$ are adjacent in V – D, similarly choose the set of vertices $u_{i,l8}$ for $l \neq 2,5$, where $l = 8,11,14,\ldots$ then the set of vertices $u_{i,l7}$, for $1 \leq i \leq n$ and $l = 8,11,14,\ldots$ are dominated. Hence the set of vertices $u_{i,l6}$, for $l = 5,8,\ldots$ and $u_{i,l7}$, for $1 \leq i \leq n$ and $l = 8,11,14,\ldots$ are adjacent in V – D. Thus Proceeding similarly, choose the set of all vertices $\{u_{i,ll}\}$, $l = 2,5,8,\ldots$ for the γ_r set D, from the path of the type $P_{it_\square} = P_{3m-1}$, for $m \geq 1$, $1 \leq i \leq n$. Let $C_1 = \{u_{i,\square 2}, u_{i,\square 5}, u_{i,\square 8}, \ldots, u_{i,\square\square}\}$ for $1 \leq i \leq n$, $l = 2,5,8,\ldots$ be the set of vertices chosen from paths of type P_{3m-1}, for $m \geq 1$for the γ_r set D.

Case (ii) If $P_{it_\square} = P_{3m}$, for $m \geq 1$, $1 \leq i \leq n$, where $l = 3,6,9,\ldots$ then the path $P_{it_\square}$ be $P_\square$, where the vertex set of $P_\square$, be $\{u_{i,\square 1}, u_{i,\square 2}, u_{i,\square 3}, \ldots, u_{i,\square\square}\}$ for $1 \leq i \leq n$, such that the vertex $u_{i,\square 1} = v_i$. Without loss of generality, choose the set of vertices $u_{i,\square 3}$, for $1 \leq i \leq n$, $l = 3,6,9,\ldots$ for theγ_r set D from the path of the type P_{3m}, for $m \geq 1$, from each u_i then the set of vertices $u_{i,\square 2}$, for $1 \leq i \leq n$, $l = 3,6,9,\ldots$ and $u_{i,\square 4}$, for $1 \leq i \leq n$, $l \neq 3$, where $l = 6,9,12,\ldots$ are dominated. Since $\{u_{i,\square 1}\} = v_i$for $1 \leq i \leq n$, $l = 3,6,9,\ldots$ are already dominated by the set of vertices $\{u_{i,\square 2}\}$, for $l = 2$, for $1 \leq i \leq n$ which is chosen for the γ_r set D from the path of the type P_{3m-1}, for $m \geq 1$. Thus the two set of vertices $\{u_{i,\square 1}\} = v_i$for $1 \leq i \leq n$, $l = 3,6,9,\ldots$ and $\{u_{i,\square 2}\}$ for $l = 3,6,9,\ldots$, $1 \leq i \leq n$ are adjacent in V – D of G. Similarly choose the set of vertices $\{u_{i,\square 6}\}$ for $1 \leq i \leq n$, $l \neq 3$, $l = 6,9,12,\ldots$for the γ_r set D, then the set $\{u_{i,\square 5}\}$ for $1 \leq i \leq n$, $l \neq 3$, $l = 6,9,12,\ldots$ are dominated. Thus the sets $\{u_{i,\square 4}\}$and $\{u_{i,\square 5}\}$ for $1 \leq i \leq n$, $l \neq 3$, $l = 6,9,12,\ldots$ are adjacent in V – D. Proceeding similarly, choose the set of all vertices $\{u_{i,\square\square}\}$,$l = 3,6,9,\ldots$, for the γ_r set D from the path of the type $P_{it_\square} = P_{3m}$, for $m \geq 1$, $1 \leq i \leq n$,$l = 3,6,9,\ldots$ Let $C_2 = \{u_{i,\square 3}, u_{i,\square 6}, u_{i,\square 9}, \ldots, u_{i,\square\square}\}$ for $1 \leq i \leq n$, $l = 3,6,9,\ldots$ be the set of all vertices chosen for the γ_r set D.

Case (iii) If $P_{it_l} = P_{3m+1}$, for $m \geq 1$, $1 \leq i \leq n$, $l = 4,7,10,\ldots$ then the path P_{it_l} be P_l. The vertex set of the path P_l, be $\{u_{i,l1}, u_{i,l2}, u_{i,l3}, \ldots, u_{i,ll}\}$ for $1 \leq i \leq n$, such that the vertex$u_{i,l1} = v_i$. Without loss of generality, choose the set of vertices as same as in the Case (ii). Thus choose the set of vertices $\{u_{i,l3}\}$, for $1 \leq i \leq n$, $l = 4,7,10,\ldots$ and $\{u_{i,l6}\}$, for $1 \leq i \leq n$, $l \neq 4$, $l = 7,10,13\ldots$ for the γ_r set D and similarly choose as same as in the case(ii) where the set of all vertices $\{u_{i,ll-1}\}$ and $\{u_{i,ll}\}$, for $l = 4,7,10,\ldots$, are considered for the γ_r set D. Let $C_3 = \{u_{i,l3}, u_{i,l6}, u_{i,l9}, \ldots, u_{i,ll-1}, u_{i,ll}\}$ for $1 \leq i \leq n$, $l = 4,7,10,\ldots$, be the set of all vertices chosen for the γ_r set D from the path of the type $P_{it_l} = P_{3m+1}$, for $m \geq 1$.

Thus the set $D = C_1 \cup C_2 \cup C_3$ where the set $D = \{u_{i,l2}, u_{i,l5}, u_{i,l8}, u_{i,l3}, u_{i,l6}, \ldots, u_{i,ll-1}, u_{i,ll}\}$, for $1 \leq i \leq n$, $l \geq 2$ forms the γ_r set D of the unicyclic graph G, with cardinality $k = \gamma_r$. Thus G is k - γ_r enresdowed for any $k = \gamma_r$.

Consider any set D_1 of cardinality $k_1 = \gamma_r + 1$, then there exists the following cases

Case (iii)(a) Consider any set D_{11} of cardinality $k_{11} = \gamma_r + 1$, where the set $D_{11} = D \cup \{u_{i,lr}\}$, $1 \le i \le n$, $2 \le l \le S_i$, $r \ge 2$, then there exists the following subcases.

Subcase (iii)(a_1) Consider a set $D_{11,1} = D \cup \{u_{i,lr}\}$, where $1 \le i \le n$, $2 \le l \le S_i$, and $r = 2$. Then the set of vertices $\{u_{i,l2}\}$ is adjacent to the $\{v_i\}$, $1 \le i \le n$. By considering any vertices $\{u_{i,l2}\}$ which belongs to the V – D set , for obtaining the restrained dominating set, there exists no isolates in the set $V - D_{11,1}$. Thus the set $D_{11,1}$ forms the restrained dominating set of cardinality $k_{11,1} = \gamma_r + 1$, containing the minimum restrained dominating set D. Hence G is $k_{11,1}$ - γ_r enresdowed.

Subcase (iii)(a_2) Consider a set $D_{11,2} = D \cup \{u_{i,lr}\}$, where $1 \le i \le n$, $2 \le l \le S_i$, and $r > 2$, then the set of vertices $\{u_{i,lr}\}$ is not adjacent with any of the vertex $\{v_i\}$, $1 \le i \le n$. Since the set of vertices $\{u_{i,lr}\}$, where $1 \le i \le n$, $2 \le l \le S_i$, and $r > 2$, belongs to the path P_l, it is adjacent only to its adjacent vertices in P_l. Thus by considering this set of vertices $\{u_{i,lr}\}$ there exists an set of isolated vertices in the set $V - D_{11,2}$ and the set $D_{11,2}$ is not a restrained dominating set of cardinality $k_{11,2} = \gamma_r + 1$. Thus G is not $k_{11,2}$ - γ_r enresdowed.

Case (iii)(b) Consider any set $D_{12} = D \cup \{v_i\}$,$1 \le i \le n$, which is of cardinality $k_{12} = \gamma_r + 1$. By considering any vertex $\{v_i\}$,$1 \le i \le n$ from the cycle C_n, $n \ge 3$, then the vertex $\{u_{i,lr}\}$ for $l = 3$ and $r = 2$, $1 \le i \le n$ form an isolate vertex in $V - D_{12}$. Therefore the set D_{12} is not a restrained dominating set of G. Hence G is not k_{12} - γ_r enresdowed for any $k_{12} = \gamma_r + 1$. Thus G is k_1 - γ_r enresdowed for any $k_1 = \gamma_r + 1$.

Consider any set D_2 of cardinality $k_2 = \gamma_r + 2$, then there exists the following cases.
Case (iii)(b_1) Consider any set $D_{21} = D \cup \{v_{i_1}, v_{i_2}\}$, $1 \le i_1, i_2 \le n$, where the vertices v_{i_1}, v_{i_2} belong to the cycle C_n, $n \ge 3$. The cardinality of the set D_{21} is $k_{21} = \gamma_r + 2$. By considering any set of vertices v_{i_1}, v_{i_2} from the cycle C_n there exists an isolates in the set $V - D_{21}$. Hence the set D_{21} is not a restrained dominating set of G. Therefore G is not k_{21} - γ_r enresdowed.

Case (iii)(b_2) Consider the set $D_{22} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, where $1 \le i_1, i_2 \le n$, $2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$.The set D_{22} is of cardinality $k_{22} = \gamma_r + 2$, then there exists the following subcases.

Subcase (iii)(b_{21}) Consider the set $D_{22,1} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, such that $1 \le i_1, i_2 \le n$, $2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$ and $l_1 = l_2$. Let $k_{22,1} = \gamma_r + 2$ be the cardinality of the set $D_{22,1}$.Thus the vertices u_{i_1,l_1r_1} and u_{i_2,l_2r_2} belong to the same path, then there exists the following subcases.

Subcase (iii)($b_{21(1)}$) Consider the set $D_{22,11} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, where $1 \le i_1, i_2 \le n$, $2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$ and $l_1 = l_2$. If the given two vertices u_{i_1,l_1r_1} and u_{i_2,l_2r_2} are adjacent in the same path. Then the set $V - D_{22,11}$ does not contain an isolate vertex. Thus the set $D_{22,11}$ forms an restrained dominating set containing the minimum restrained dominating set. Hence G is $k_{22,11}$ - γ_r enresdowed for any $k_{22,11} = \gamma_r + 2$.

Subcase (iii)($b_{21(2)}$) Consider the set $D_{22,12} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, where $1 \le i_1, i_2 \le n$, $2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$ and $l_1 = l_2$. If the vertices u_{i_1,l_1r_1} , u_{i_2,l_2r_2} are not adjacent in the same path, since these vertices belong to a path, a set of vertices adjacent to u_{i_1,l_1r_1} , u_{i_2,l_2r_2} in the path forms a set of isolates. Thus the set of isolate vertices exists in the set $V - D_{22,12}$. Thus the set $D_{22,12}$ is not a restrained dominating set of cardinality $k_{22,12} = \gamma_r + 2$. Hence G is not $k_{22,12}$ - γ_r enresdowed.

Subcase (iii)(b_{22}) Consider the set $D_{22,2} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, such that $1 \le i_1, i_2 \le n$, $2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$ and $l_1 \ne l_2$, then the following subcases exists.

Subcase (iii)($b_{22(1)}$) Consider the set $D_{22,21} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}\}$, where $1 \le i_1, i_2 \le n$, $2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$ and $i_1 = i_2$, then there exists the following subcases.

Subcase (iii)($b_{22(11)}$) Consider the set $D_{22,211}$ = D ∪ {u_{i_1,l_1r_1} , u_{i_2,l_2r_2}},where $1 \le i_1, i_2 \le n$, $2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$ and $i_1 = i_2$, such that the vertices u_{i_1,l_1r_1} and u_{i_2,l_2r_2} belong to the different paths and they are attached to the same v_i, $1 \le i \le n$ of C_n, n≥ 3, which result in non – existence of an isolate vertex in the set V –$D_{22,211}$. Hence the set $D_{22,211}$ forms a restrained dominating set containing the γ_r set D, with cardinality $k_{22,211} = \gamma_r + 2$. Therefore G is $k_{22,211}$ - γ_r enresdowed.

Subcase (iii)($b_{22(12)}$) Consider the set $D_{22,212}$ = D ∪ {u_{i_1,l_1r_1} , u_{i_2,l_2r_2}},where $1 \le i_1, i_2 \le n$, $2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$ and $i_1 = i_2$, then the vertices u_{i_1,l_1r_1} , u_{i_2,l_2r_2} which belong to different paths are not adjacent to any vertex v_i, $1 \le i \le n$ of C_n, n≥ 3 ,then there exists isolates in V –$D_{22,212}$. Thus the set $D_{22,212}$ is not a restrained dominating set of cardinality $k_{22,212} = \gamma_r + 2$. Therefore G is not $k_{22,212}$ - γ_r enresdowed for any cardinality $k_{22,212}$ = γ_r + 2.

Subcase (iii)($b_{22(2)}$) Consider the set $D_{22,22}$ = D ∪ {u_{i_1,l_1r_1} , u_{i_2,l_2r_2}}, where $1 \le i_1, i_2 \le n$, $2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$ and $i_1 \ne i_2$. The vertices u_{i_1,l_1r_1} and u_{i_2,l_2r_2} belong to the different paths, where the paths are attached to the differentv_i, $1 \le i \le n$.

Subcase (iii)($b_{22(21)}$) Consider the set $D_{22,221}$ = D ∪ {u_{i_1,l_1r_1}, u_{i_2,l_2r_2}}, $1 \le i_1, i_2 \le n$,$2 \le l_1, l_2 \le S_{i1}$, $r_1, r_2 \ge 2$ and $i_1 \ne i_2$, such that the verticesu_{i_1,l_1r_1} and u_{i_2,l_2r_2} are adjacent to the vertex v_i, $1 \le i \le n$. Then there exists no isolate in the set V – $D_{22,221}$. Hence the set $D_{22,221}$ forms a restrained dominating set containing the γ_r set D of G. The cardinality of the set $D_{22,221}$ be $k_{22,221} = \gamma_r + 2$. Hence G is $k_{22,221}$ - γ_r enresdowed.
Subcase (iii)($b_{22(22)}$) Consider the set $D_{22,222}$ = D ∪ {u_{i_1,l_1r_1} , u_{i_2,l_2r_2}}, where the vertices u_{i_1,l_1r_1} , u_{i_2,l_2r_2} which belong to different paths of G and are not adjacent to v_i, $1 \le i \le n$, then there exists isolated vertices in the set V – $D_{22,222}$. Thus the set $D_{22,222}$ is not a restrained dominating set of G with cardinality $k_{22,222} = \gamma_r + 2$. Hence G is not $k_{22,222}$ - γ_r enresdowed.

Case (iv) Consider the set D_{23} = D ∪{v_{i_1},$u_{i,lr}$},$1 \le i_1, i \le n$, $2 \le l \le S_{i1}$, r > 2 and {v_i}, $1 \le i \le n$, be any vertex of the unicycle C_n, n≥ 3. By choosing any vertex from the cycle, the set of vertices adjacent to the vertex v_i form an isolated set of vertices, where the cardinality of the set D_{23} be $k_{23} = \gamma_r + 2$. Thus the set V – D_{23} contains the isolated vertices and the set D_{23} is not a restrained dominating set. Thus G is not k_{23} - γ_r enresdowed and G is k_2 - γ_r enresdowed for any $k_2 = \gamma_r + 2$.

Case (v) Consider any set D_3 of cardinality $k_3 = \gamma_r + 3$, then the following subcases exists.
Case (v)(a) Consider the set D_{31},where D_{31} = D ∪{v_{i_1}, v_{i_2},v_{i_3}}, $1 \le i_1, i_2, i_3 \le n$. By choosing the vertices v_{i_1}, v_{i_2},v_{i_3} from the unicycle C_n, n≥ 3 for the set D_{31}, the set V – D_{31} contains a set of isolated vertices . Thus the set D_{31} is not a restrained dominating set of cardinality $k_{31} = \gamma_r + 3$. Hence G is not k_{31} - γ_r enresdowed.
Case (v)(b) Consider the set D_{32}, where the set D_{32} = D ∪ {u_{i_1,l_1r_1} , u_{i_2,l_2r_2}, u_{i_3,l_3r_3}}, $1 \le i_1, i_2, i_3 \le n$, $2 \le l_1, l_2, l_3 \le S_{i1}$,$r_1, r_2, r_3 \ge 1$ of cardinality $k_{32} = \gamma_r + 3$, then there exists the following subcases.

Subcase (v)(b_1) Consider the set $D_{32,1}$, where $D_{32,1}$ = D ∪ {u_{i_1,l_1r_1} , u_{i_2,l_2r_2}, u_{i_3,l_3r_3}},$1 \le i_1, i_2, i_3 \le n$,$2 \le l_1, l_2, l_3 \le S_{i1}$, $r_1, r_2, r_3 \ge 1$ and $r_1 = r_2 = r_3 \ne 1$. If these set of vertices u_{i_1,l_1r_1} , u_{i_2,l_2r_2} and u_{i_3,l_3r_3} are adjacent either to the same vertex v_i or different v_i, $1 \le i \le n$, then there exists no isolates in the set V – $D_{32,1}$. Thus the set $D_{32,1}$ forms a restrained dominating set containing the minimum restrained dominating set of cardinality $k_{32,1} = \gamma_r$ + 3. Hence G is $k_{32,1}$ - γ_r enresdowed.

Subcase (v)(b_2) Consider the set $D_{32,2}$ = D ∪ {u_{i_1,l_1r_1} , u_{i_2,l_2r_2}, u_{i_3,l_3r_3}}, $1 \le i_1, i_2, i_3 \le n$, $2 \le l_1, l_2, l_3 \le S_{i1}$, $r_1, r_2, r_3 > 1$, where the set of vertices are chosen from the same path of the type P_{3m-1}, for m ≥ 1, where the paths

P_{3m-1} are attached to same v_i or different v_i, $1 \le i \le n$ then there exists isolates in the set $V - D_{32,2}$. The set $D_{32,2}$ is not a restrained dominating set of cardinality $k_{32,2} = \gamma_r + 3$. Hence G is not $k_{32,2}$ - γ_r enresdowed.

Case (vi) Consider the set,D_{33} where $D_{33} = D \cup \{v_{i_1}, v_{i_2}, u_{i,lr}\}$, $1 \le i_1, i_2, i \le n$, $2 \le l \le S_i$, $r > 1$.The cardinality of the set D_{33} be $k_{33} = \gamma_r + 3$. Since the vertices of the cycle are chosen, the set D_{33} is not a restrained dominating set. Hence G is not k_{33} - γ_r enresdowed.

Case (vii) Consider the set $D_{34} = D \cup \{u_{i_1,l_1r_1}, u_{i_2,l_2r_2}, v_i\}$,$1 \le i_1, i_2, i \le n$, $2 \le l_1, l_2, \le S_{i1}$, $r_1, r_2 \ge 1$. Thus the existence of the vertex v_i of the cycle C_n, n$\ge$ 3 in the set D_{34} results in the existence of isolates in the set $V - D_{34}$. Thus the set D_{34} is not a restrained dominating set of cardinality $k_{34} = \gamma_r + 3$. Hence G is not k_{34} - γ_r enresdowed.

Proceeding similarly, consider any set D_4 of cardinalit $k_4 = n + \left| \bigcup^{n}_{\substack{i=1 \\ 2 \le j \le S_i}} P_{it_j} - S_i \right| - 1$, then the set D_4 is not a restrained dominating set of G since there exists an isolate vertices in the set $V - D_4$. Hence G is not k_4 - γ_r enresdowed. Without loss of generality, consider a set D_5 of cardinality k_5 , where the cardinality k_5 is the union of the cardinality of the set of all vertices of cycle C_n, n$\ge$ 3 and the cardinality of the set of all vertices in each path $\{P_{it_j}\}$, $1 \le i \le n$ and $2 \le j \le S_i$, except the set of all vertices $\{v_i\}$,$1 \le i \le n$ of the cycle C_n . Therefore the cardinality k_5 is given by n+ $\left| \bigcup^{n}_{\substack{i=1 \\ 2 \le j \le S_i}} P_{it_j} - S_i \right|$. Hence G is k - γ_r enresdowed for any k , where $\gamma_r \le k \le$ n+ $\left| \bigcup^{n}_{\substack{i=1 \\ 2 \le j \le S_i}} P_{it_j} - S_i \right|$

REFERENCES

1. *Anders Sune Pedersen ,PrebenDahlVestergaard,"The number of independent sets in unicyclic graphs"Discrete Applied Mathematics,Volume 152, Issues 1–3, 1 November 2005, pp. 246-256.*
2. *JoannaRaczek." On Domination Multisubdivision Number OfUnicyclic Graphs",Opuscula Math. 38, no. 3 (2018), pp 409–425 https://doi.org/10.7494/OpMath.2018.38.3.409.*
3. *Omidi. G. R. On Integral Graphs with Few Cycles. Graphs Combin., 25 (2009), 841–849.*
4. *Rodrigo O. Braga, Virg ́ınia M. Rodrigues, VilmarTrevisan, Locating Eigenvalues Of Unicyclic Graphs, Appl. Anal.Discrete Math. 11 (2017), 273–298.*
5. *Sumathi.P, Esther Felicia.R, Enresdowed graphs II, Global Journal of Pure and Applied Mathematics , Volume 13, Number 1, 2017, pp 229 – 232 .*
6. *Teresa W.Haynes,Stephen .Hedetniemi,Peter J. Slater, Fundamentals of Domination in Graphs,Marcel Dekker,INC.New York,1998.*
7. *Zhibin Du,Spectral Properties Of A Class Of Unicyclic Graphs,Journal of Inequalities and Applications. 2017; 2017(1): 96.doi: 10.1186/s13660-017-1367-2.*